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ADJUSTMENT CURVES FOR BINARY RESPONSES ASSOCIATED TO STOCHASTIC PROCESSES

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Adjustment curves for binary responses associated to stochastic processes

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Abstract

We consider a class $\mathcal{S}$ of stochastic processes $X := \{X(t)\}_{t\in[0,T]}$ whose realizations $x := x(t)$ $(t \in [0,T])$ are real continuous piecewise linear functions satisfying a particular geometric condition. Let $\mathcal{R}$ be the family of all binary responses $Y, Y \in \{bad, good\}$, associated to a process X in $\mathcal{S}$. Basing on data arising from a continuous phenomenon which can be simulated by a couple $(X,Y) \in \mathcal{S} \times \mathcal{R}$, we introduce the notion of *adjustment curve for the binary response* Y *of the process* X, that is a decreasing function $\gamma_a : [0,T] \to [0,1]$ which gives the probability that a new realization x of X is adjustable at the time $t \in [0,T]$. For real industrial processes, which can be modelized by $(X,Y) \in \mathcal{S} \times \mathcal{R}$, our model can be used for monitoring and predicting the quality of the product.

KEY WORDS: Functional Data; Random Multiplicative Cascade; Adjustment Curve; Stochastic Process.

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1 Introduction, notations and definitions

In many real applications we are faced with a continuous phenomenon evolving in a certain interval of time say $[0, T]$, and resulting in an outcome not observable before the completion of the process itself. Such outcome, in accordance with some target-value, in the simplest case can be expressed as negative or positive, bad or good and so on. We restrict our attention to a class of such phenomenons each of one can be represented by a stochastic process whose realizations are real continuous functions with $\{x(0) : x \in X\}$ finite, linear on the intervals $[t_j, t_{j+1}]$ with $t_j = j \cdot \frac{T}{S}$ for $j = 0, \ldots, S-1$ and satisfying a particular geometric condition. In the following $\mathcal{S}$ will denote the class of such stochastic processes, $X := \{X(t)\}_{t \in [0,T]}$ a stochastic process in $\mathcal{S}$, $x := x(t)$ $(t \in [0, T])$ a realization of X and $\{x_0^i : i = 1, \ldots, \text{Ł}\} := \{x(0) : x \in X\}$. Moreover $\mathcal{R}$ will denote the class of all binary responses $Y, Y \in \{bad, good\}$, associated to X.

In the following $\mathbb{N}$ and $\mathbb{R}$ stand for the set of all natural and real numbers, respectively. For a finite set $S \subset \mathbb{R}$, $|S|$, $\min S$ and $\max S$ denote its cardinality, minimum and maximum, respectively. Moreover $\chi_A : [0, 1] \to \mathbb{R}$ denotes the characteristic function of a subset $A \subset [0, 1]$.

Definition 1 *Let A, B be two non-empty finite subsets of $\mathbb{R}$, $\epsilon > 0$ and $m, n \in \mathbb{N}$. The sets A and B are called $\epsilon - (m, n)$ separated if there exist sets $\tilde{A} \subset A$ and $\tilde{B} \subset B$, with cardinalities $\left|\tilde{A}\right| = m$ and $\left|\tilde{B}\right| = n$, such that*

$$\min\left(A \setminus \tilde{A}\right) \geq \max\left(B \setminus \tilde{B}\right) \ \ or \ \ \min\left(B \setminus \tilde{A}\right) \geq \max\left(B \setminus \tilde{B}\right) \qquad (1)$$

and

$$\left| \frac{|A|}{|A| + |B|} - \frac{\left|A \setminus \tilde{A}\right|}{\left|A \setminus \tilde{A}\right| + \left|B \setminus \tilde{B}\right|} \right| \leq \epsilon. \qquad (2)$$

If we can choose $\tilde{A} = \tilde{B} = \emptyset$ then $\min(A) \geq \max(B)$ or $\min(B) \geq \max(A)$ and the sets A and B are called *separated*.

The starting point is the matrix of real numbers (functional data) (for an exhaustive account to functional data analysis see Ramsay and Silverman, 2002, 2005)

$$FD := \begin{bmatrix} x^1(0) & \dots & x^1(t_j) & \dots & x^1(T) \\ \vdots & & \vdots & & \vdots \\ x^i(0) & \dots & x^i(t_j) & \dots & x^i(T) \\ \vdots & & \vdots & & \vdots \\ x^L(0) & \dots & x^L(t_j) & \dots & x^L(T) \end{bmatrix} \quad (t_j = j \cdot \frac{T}{S} \text{ for } j = 0, \dots, S)$$

and the vector

$$R := \begin{bmatrix} r^1 \\ \vdots \\ r^i \\ \vdots \\ r^L \end{bmatrix}$$

where $r^i \in \{bad, good\}$ is the outcome associated to the row $(x^i(0) \dots x^i(t_j) \dots x^i(T))$ for $i = 1, \dots, L$. We suppose that the data FD and R arise from a continuous phenomenon which can be simulated by a couple $(X, Y) \in \mathcal{S} \times \mathcal{R}$. X and Y are realized via random multiplicative cascades which depend on some constants (obtainable by the data FD and R) and some real positive parameters obtained by a Monte-Carlo method.

Set

$$B = \left\{ x^i(T) : i = 1, \dots, L \ and \ r^i = bad \right\}$$

and

$$G = \left\{ x^i(T) : i = 1, \dots, L \ and \ r^i = good \right\}.$$

In many real cases the sets B and G are not separated. However, if the sets B and G are $\epsilon - (m, n)$ separated with ϵ, m, n small enough, we can reduce, by a proper smoothing of the data, to the separated case.

Remark 2 In some real cases in the vector

$$R := \begin{bmatrix} r^1 \\ \vdots \\ r^i \\ \vdots \\ r^L \end{bmatrix}$$

we could have

$$r^i \in \{bad, good, undecidible\}\,. \tag{3}$$

The set

$$I = \left\{x^i(T) : i = 1, \ldots, L \ \ and \ r^i = undecidible\right\}$$

is not $\epsilon - (m, n)$ separated by the set $B = \{x^i(T) : i = 1, \ldots, L \ \ and \ r^i = bad\}$ (or by the set $G = \{x^i(T) : i = 1, \ldots, L \ \ and \ r^i = good\}$) if we can not choose the numbers ϵ, m, n – without modifying the phenomenon in a significant way – to get I and B (or I and G) $\epsilon - (m, n)$ separated. However, if the sets B and G are $\epsilon - (m, n)$ separated with a proper smoothing of the data, by deleting in FD the rows

$$\left\{\left(x^i\left(0\right), \ldots, x^i\left(T\right)\right) : i = 1, \ldots, L \ \ and \ r^i = undecidible\right\}$$

Y becomes a binary outcome.

The main aim of this paper is to introduce the notion of *adjustment curve for the binary response Y of the process X*, that is a decreasing function $\gamma_a : [0, T] \rightarrow [0, 1]$ which gives the probability that a new realization x of X is adjustable at the time $t \in [0, T]$. The curve γ_a is an important tool from a practical point of view. For example, for real industrial processes which can be modelized by $(X, Y) \in \mathcal{S} \times \mathcal{R}$, our model can be used for monitoring and predicting the quality of the product (several examples where prediction is useful for controlling process can be found in Box and Kramer 1992, Ratcliffe et al. 2002, Ratcliff et al 2002, Kesavan et al. 2000).

A multiplicative cascade is a single process that fragments a set into smaller and smaller components according to a fixed rule, and at the same time fragments the *measure* of components by another rule. It is well known the central role that the multiplicative cascades play in the theory of multifractal measures (see, for example, Peitgan et al., 2004). In Sec.2 we define a random multiplicative cascade generating a multifractal measure μ on the family of all dyadic subintervals of the unit interval $[0,1]$. This measure μ is recursively generated with the cascade that is schematically depicted in Fig.1 at the end of Sec.2. In Sec.3 we describe how modelize – via the random multiplicative cascade introduced in Sec.2 – a real phenomenon by a couple $(X,Y) \in \mathcal{S} \times \mathcal{R}$. In Sec.4 we give the definition of adjustment curve for the binary response Y of a process X. Finally, in Sec.5 we illustrate an application of our method to the Danone Vitapole kneading process already analyzed in previous works (see Costanzo er al. 2006, Preda et al. 2007, Saporta et al. 2008).

2 The multiplicative cascade

We start introducing the following rule.

Let $\alpha, \beta \in (0,+\infty), k \in \mathbb{N}, q^k \in (0,1)$ and $m_i^k \in (0,1]$ $\left(i = 1,...,2^k\right)$ such that $\sum_{i=1}^{2^k} m_i^k = 1$. Put $m_0^k := 0$ and

$$J_i^k := \begin{cases} \left[\sum_{j=0}^{i-1} m_j^k, \sum_{j=0}^{i} m_j^k\right) & \text{if } i = 1,...,2^k - 1, \\ \left[\sum_{j=0}^{2^k-1} m_j^k, 1\right] & \text{if } \; i = 2^k. \end{cases}$$

Let pos_k denote the step-function:

$$pos_k(x) := \sum_{i=1}^{2^k} i \chi_{J_i^k}(x) \quad (x \in [0,1]).$$

If x_k is a random generated number in the interval $[0,1]$.We define the vectors

$$M^{k}=\left(m_{1}^{k},...,m_{2^{k}}^{k}\right);$$

$$Q_{pos_{k}(x_{k})}^{k+1}=\left(q_{pos_{k}(x_{k})}^{k+1},1-q_{pos_{k}(x_{k})}^{k+1}\right),$$

where, if $k=0$

$$q_{pos_{0}(x_{0})}^{1}:=q^{0},$$

if $k\geq 1$

$$q_{pos_{k}(x_{k})}^{k+1}:=\left\{\begin{array}{l} q^{k}+\left(\min\left\{q^{k},1-q^{k}\right\}\right)^{\alpha}\frac{1}{2^{\left(pos_{k}(x_{k})^{\beta}\right)}}, \\ \text{if } 1\leq pos_{k}\left(x_{k}\right)\leq 2^{k-1}; \\ q^{k}-\left(\min\left\{q^{k},1-q^{k}\right\}\right)^{\alpha}\frac{1}{2^{\left(\left(2^{k}-pos_{k}(x_{k})+1\right)^{\beta}\right)}}, \\ \text{ if } 2^{k-1}<pos_{k}\left(x_{k}\right)\leq 2^{k}. \end{array}\right.$$

Let $\left(Q_{pos_{k}(x_{k})}^{k+1}\right)^{T}$ be the transpose of the vector $\left(Q_{pos_{k}(x_{k})}^{k+1}\right)$.We consider the matrix product

$$\left(Q_{pos_{k}(x_{k})}^{k+1}\right)^{T}*M^{k}=\left(\begin{array}{ccc} m_{1}^{k}q_{pos_{k}(x_{k})}^{k+1} & \cdots & m_{2^{k}}^{k}q_{pos_{k}(x_{k})}^{k+1} \\ m_{1}^{k}\left(1-q_{pos_{k}(x_{k})}^{k+1}\right) & \cdots & m_{2^{k}}^{k}\left(1-q_{pos_{k}(x_{k})}^{k+1}\right) \end{array}\right).$$

Put

$$m_{i;pos_{k}(x_{k})}^{k+1}:=\left\{\begin{array}{l} m_{i}^{k}q_{pos_{k}(x_{k})}^{k+1} \text{ for } i=1,...,2^{k} \\ m_{i-2^{k}}^{k}\left(1-q_{pos_{k}(x_{k})}^{k+1}\right) \text{ for } i=2^{k}+1,...,2^{k+1}. \end{array}\right. \tag{4}$$

We note that

$$q_{pos_{k}(x_{k})}^{k+1}\in(0,1),m_{i;pos_{k}(x_{k})}^{k+1}\in(0,1]\quad\left(i=1,...,2^{k+1}\right),$$

and

$$\sum_{i=1}^{2^{k+1}}m_{i;pos_{k}(x_{k})}^{k+1}=1.$$

Therefore the above procedure can be iterated.

In the following we denote by $\mathcal{I}$ the family of all dyadic subintervals

$$I_{i}^{k}=\left[\frac{i-1}{2^{k}},\frac{i}{2^{k}}\right]:\left(i,k\in\mathbb{N};i=1,...,2^{k}\right)$$

of the unit interval $[0,1]$.

Let $q^{1}_{pos_0(x_0)} := q^0 \in (0,1)$ be arbitrarily fixed. Our cascade starts ($k = 0$) with a uniformly distributed unit mass, $m^0_1 = 1$, on the interval $I^0_1 = [0,1]$. At the stage ($k = 1$) our cascade splits the interval I^0_1 into the intervals $I^1_i\,(i = 1,2)$, and at same time uniformly distributes - according to the eq. (4) - the mass m^0_1, by distributing a fraction $m^1_{1;pos_0(x_0)} := q^1_{pos_0(x_0)}$ uniformly on I^1_1, and the remaining fraction $m^1_{2,pos_0(x_0)} = 1 - m^1_{1,pos_0(x_0)}$ uniformly on I^1_2. A this stage, I^1_1 carries the measure $\mu\left(I^1_1\right) = m^1_{1;pos_0(x_0)}$ and I^1_2 carries the measure $\mu\left(I^1_2\right) = m^1_{2;pos_0(x_0)}$. In this process, because $\mu\left(I^0_1\right) = \mu\left(I^1_1\right) + \mu\left(I^1_2\right) = 1$, the original measure of the unit interval is conserved; the μ's appear like probabilities, and one says that μ is a probability measure. At the next stage ($k = 2$) our cascade splits the unit interval into the intervals $I^2_i\,(i = 1,...,4)$, and at same time uniformly distributes the masses $m^1_{1;pos_0(x_0)}, m^1_{2,pos_0(x_0)}$ - according the eq. (4) - over the intervals $I^2_i\,(i = 1,...,4)$. Precisely, let x_1 be a random number generated in the interval $[0,1]$. This second stage of the cascade yields:

$$\begin{cases} m^2_{1,pos_1(x_1),pos_0(x_0)} = m^1_{1;pos_0(x_0)} q^2_{pos_1(x_1),pos_0(x_0)}, \\ m^2_{2,pos_1(x_1),pos_0(x_0)} = m^1_{2;pos_0(x_0)} q^2_{pos_1(x_1),pos_0(x_0)}, \\ m^2_{3,pos_1(x_1),pos_0(x_0)} = m^1_{1pos_0(x_0)}(1 - q^2_{pos_1(x_1),pos_0(x_0)}), \\ m^2_{4,pos_1(x_1),pos_0(x_0)} = m^1_{2pos_0(x_0)}\left(1 - q^2_{pos_1(x_1),pos_0(x_0)}\right), \end{cases}$$

where

$$q^2_{pos_1(x_1),pos_0(x_0)} = \begin{cases} q^1_{pos_0(x_0)} + \left(\min\left\{q^1_{pos_0(x_0)}, 1 - q^1_{pos_0(x_0)}\right\}\right)^{\alpha} \frac{1}{2^{\left((x_1)^{\beta}\right)}}, \\ \text{if } pos_1\left(x_1\right) = 1, \\ q^1_{pos_0(x_0)} - \left(\min\left\{q^1_{pos_0(x_0)}, 1 - q^1_{pos_0(x_0)}\right\}\right)^{\alpha} \frac{1}{2^{\left((2-pos_1(x_1)+1)^{\beta}\right)}}, \\ \text{if } pos_1\left(x_1\right) = 2. \end{cases}$$

At the stage k^{th} ($k \geq 2$) of the cascade the unit interval is split into the intervals $I^k_i\left(i = 1,...,2^k\right)$, and at same time the masses $m^{k-1}_{i;pos_{k-2}(x_{k-2}),...,pos_0(x_0)}$ $\left(i = 1,...,2^{k-1}\right)$ are uniformly distributed, according to the eq. (4), over the intervals $I^k_i\left(i = 1,...,2^k\right)$. Let x_{k-1} be a random number generated in the interval $[0,1]$ .We obtain:

$$m^{k}_{i;pos_{k-1}(x_{k-1}),\dots,pos_0(x_0)} := \begin{cases} m^{k-1}_{i;pos_{k-2}(x_{k-2}),\dots,pos_0(x_0)} q^{k}_{pos_{k-1}(x_{k-1}),\dots,pos_0(x_0)} \\ \left(i=1,\dots,2^{k-1}\right), \\ m^{k-1}_{i-2^{k-1};pos_{k-2}(x_{k-2}),\dots,pos_0(x_0)} q^{k}_{pos_{k-1}(x_{k-1}),\dots,pos_0(x_0)} \\ \left(i=2^{k-1}+1,\dots,2^{k}\right), \end{cases}$$

where

$$q^{k}_{pos_{k-1}(x_{k-1}),\dots,pos_0(x_0)} = \begin{cases} q^{k-1}_{pos_{k-2}(x_{k-2}),\dots,pos_0(x_0)} + \\ + \left(\min\left\{q^{k-1}_{pos_{k-2}(x_{k-2}),\dots,pos_0(x_0)}, 1-q^{k-1}_{pos_{k-2}(x_{k-2}),\dots,pos_0(x_0)}\right\}\right)^{\alpha} \\ \frac{1}{2^{\left(pos_{k-1}(x_{k-1})^{\beta}\right)}}, \text{ if } 1 \leq pos_{k-1}\left(x_{k-1}\right) \leq 2^{k-2}; \\ q^{k-1}_{pos_{k-2}(x_{k-2}),\dots,pos_0(x_0)} + \\ - \left(\min\left\{q^{k-1}_{pos_{k-2}(x_{k-2}),\dots,pos_0(x_0)}, 1-q^{p-1}_{pos_{k-2}(x_{k-2}),\dots,pos_0(x_0)}\right\}\right)^{\alpha} \\ \frac{1}{2^{\left(\left(2^{k-1}-pos_{k-1}(x_{k-1})+1\right)^{\beta}\right)}}, \text{ if } 2^{k-2} < pos_{k-1}\left(x_{k+1}\right) \leq 2^{k-1}. \end{cases}$$

We have that $q^{k}_{pos_{k-1}(x_{k-1}),\dots,pos_0(x_0)} \in (0,1)$, $m^{k}_{i;pos_{k-1}(x_{k-1}),\dots,pos_0(x_0)} \in (0,1]$ $\left(i=1,\dots,2^{k}\right)$and $\sum\limits_{i=1}^{2^k} m^{k}_{i;pos_{k-1}(x_{k-1}),\dots,pos_k(x_k)} = 1$.

Set

$$\mu\left(I^{k}_{i}\right) := m^{k}_{i;pos_{k-1}(x_{k-1}),\dots,pos_0(x_0)} \quad \left(i=1,\dots,2^{k}\right).$$

We observe that

$$m^{k-1}_{i;pos_{k-2}(x_{k-2}),\dots,pos_0(x_0)} = m^{k}_{i;pos_{k-1}(x_{k-1}),\dots,pos_0(x_0)} + \\ + m^{k}_{2^{k-1}+i;pos_{k-1}(x_{k-1}),\dots,pos_0(x_0)} \quad \left(i=1,\dots,2^{k-1}\right),$$

and therefore

$$\mu\left(I^{k-1}_{i}\right) = \mu\left(I^{k}_{i}\right) + \mu\left(I^{k}_{2^{k-1}+i}\right) \quad \left(i=1,\dots,2^{k-1}\right).$$

Then the above cascade (schematically depicted in fig. 1) produces the multifractal measure μ which attributes masses according to the eq. (4) to the family $\mathcal{I}$ of all dyadic subintervals of the unit interval $[0,1]$.

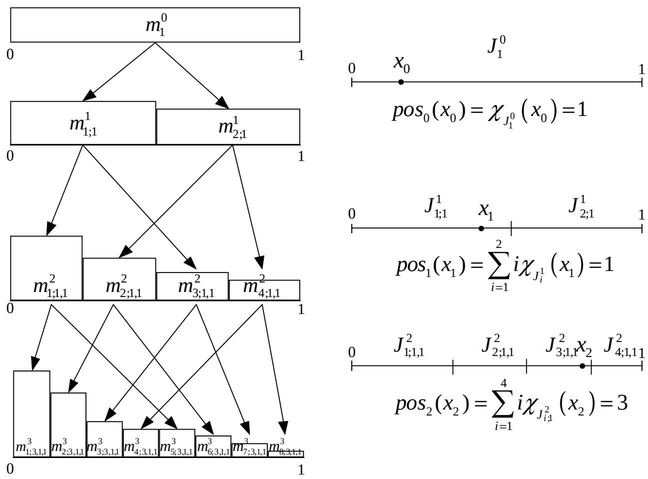

Figure 1: The first four stages of the multiplicative cascade

3 The process X and the associated binary response Y

Let

$$FD := \begin{bmatrix} x^1(0) & \dots & x^1(t_j) & \dots & x^1(T) \\ \vdots & & \vdots & & \vdots \\ x^i(0) & \dots & x^i(t_j) & \dots & x^i(T) \\ \vdots & & \vdots & & \vdots \\ x^L(0) & \dots & x^L(t_j) & \dots & x^L(T) \end{bmatrix} \quad \left(t_j = j \cdot \frac{T}{S} \text{ for } j = 0, \dots, S\right)$$

be the functional data which arise from a continuous phenomenon and let

$$R := \begin{bmatrix} r^1 \\ \vdots \\ r^i \\ \vdots \\ r^L \end{bmatrix} \quad r^i \in \{bad, good\}$$

be the outcome associated to the row $(x^i(0) \dots x^i(t_j) \dots x^i(T))$ for $i = 1, \dots, L$.

Set

$$B := \left\{ x^i(T) : i = 1, \dots, L \text{ and } r^i = bad \right\}$$

and

$$G := \left\{ x^i(T) : i = 1, \dots, L \text{ and } r^i = good \right\}.$$

We assume that B and G are separated with $\min(G) > \max(B)$. In this section we describe how to modelize – via the random multiplicative cascade introduced in Sec.2 – the real phenomenon by a couple $(X, Y) \in \mathcal{S} \times \mathcal{R}$.

Step 1. We use the data FD and R in order to define the following constants:

$$q^0 := \frac{|G|}{|G| + |B|} \text{ and } 1 - q^0 = \frac{|B|}{|G| + |B|};$$

$$a := \min_{i=1,\ldots,L} \left(x^i\left(T\right) - x^i\left(0\right)\right) - h \text{ and } b := \max_{i=1,\ldots,L} \left(x^i\left(T\right) - x^i\left(0\right)\right) + h,$$

where h is a corrective term (the *elevation-correction term*);

$$p := \frac{\max\limits_{i=1,\ldots,L} \left(x^i\left(T\right) - x^i\left(0\right)\right) - \min\limits_{i=1,\ldots,L} \left(x^i\left(T\right) - x^i\left(0\right)\right)}{\delta},$$

where

$$\delta := \frac{\sum\limits_{i=1}^{L} \sum\limits_{j=1}^{S} |x^i(t_j) - x^i(t_{j-1})|}{LS}.$$

Let

$$s_D^i : [0, T] \to \mathbb{R} \ \ (i = 1, ..., L)$$

be the piecewise linear functions whose node-sets are

$$N^i := \left\{ \left(j, x^i(t_j)\right) : t_j = j \cdot \frac{T}{S} \text{ for } j = 0, \ldots, S \right\} \quad (i = 1, ..., L)\,.$$

Set

$$b_D(j) := \max\left\{x^i(t_j) : i = 1, \ldots, L \ \textit{and}\ r^i = \textit{bad}\right\} \quad (j = 0, \ldots, S)$$

and

$$g_D(j) := \min\left\{x^i(t_j) : i = 1, \ldots, L \ \textit{and}\ r^i = \textit{good}\right\} \quad (j = 0, \ldots, S).$$

Now we introduce the definition of the adjustment curve $\gamma_{a,D} : [0, T] \to [0, 1]$ for the binary outcome R of the functional data FD.

Definition 3 *Let $i \in \{1, \dots, L\}$ and $r^i = bad$ or $r^i = good$. The piecewise linear interpolant s^i_D is called adjustable at the time $t \in [0, T]$ (for short $t-$adjustable) if there exists $t_j \geq t$ with $j \in \{0, 1, \dots, S\}$ such that*

$$s^i_D(t_j) \geq g_D(t_j) \text{ or } s^i_D(t) \leq b_D(t_j).$$

Definition 4 *The adjustment curve $\gamma_{a,D} : [0, T] \to \mathbb{R}$ for the binary outcome R of the functional data FD is the function*

$$\gamma_{a,D}(t) = \frac{|\{s^i_D : i = 1, \dots, L \text{ and } s^i_D \text{ is } t - adjustable\}|}{L} \quad (t \in [0, T])$$

Step 2. Let (α, β) be a pair of random generated numbers in the square $[10^{-1}, 10] \times [10^{-1}, 10]$. Let

$$V_p := \left\{(n_0, n_1, ..., n_p) \in \mathbb{N}^{p+1} : 1 \leq n_k \leq 2^k \; for \;\; k = 0, 1, .., p\right\}.$$

For given α, β and $q \in (0, 1)$ our random multiplicative cascade truncated at the stage p generates a vector

$$(n_0, n_1, ..., n_p) \in V_p.$$

We define the function

$$\varphi_{a,b,p} : V_p \to \mathbb{R}^{p+1}$$

as follows:

$$\varphi_{a,b,p}\left(n_0, n_1, ..., n_p\right) = \left(y_0, y_1, ..., y_p\right),$$

where $y_0 = 0$ and the coordinates y_k, for $k = 1, ..., p$, are recursively defined by the following formula:

$$y_k = \begin{cases} y_{k-1} + \frac{a}{p} + \frac{b-a}{2p}\frac{1}{2^{k-1}}\left(n_k - 1 + z_k\right) & if \;\; 1 \leq n_k \leq 2^{k-1}, \\ y_{k-1} + \frac{b}{p} - \frac{b-a}{2p}\frac{1}{2^{k-1}}\left(2^k - n_k + z_k\right) & if \;\; 2^{k-1} < n_k \leq 2^k, \end{cases}$$

where z_k is a random number generated in the interval $[0, 1]$. Now we give the notion of experiment of size L and length $p + 1$. We introduce the following terminology. A realization of the multiplicative cascade truncated at the stage p^{th} $(p \geq 1)$ is called a *proof of length* $p+1$ and the vector

$$(0, y_1, ..., y_p) = \varphi_{a,b,p}(n_0, n_1, ..., n_p)$$

its *result*. For each $\bar{y} \in \mathbb{R}$ the vector

$$(\bar{y}, \bar{y} + y_1, ..., \bar{y} + y_p)$$

is called a result of the proof at $\bar{y}$.

For each $i \in \{1, \ldots, L\}$, if $r^i = good$ (respectively $r^i = bad$) we start the cascade with $q = q^0$ (respectively $q = 1 - q^0$) by truncating it at the stage p. Then for each $i = 1, \ldots, L$ we consider the results $(0, y_1^i, ..., y_p^i) = \varphi_{a,b,p}\left(n_0^i, n_1^i, ..., n_p^i\right)$ of the above proofs and the results $(x^i(0), x^i(0)+y_1^i, ..., x^i(0)+y_p^i)$ at $x^i(0)$. The set

$$E_p = \left\{(x^i(0), x^i(0) + y_1^i, ..., x^i(0) + y_p^i),\ i = 1, ..., L\right\}$$

is called an *experiment of size* L *and length* $p+1$.

We consider the matrix $SFD_{\alpha,\beta}$ of experiment's data (*simulated functional data*):

$$SFD_{\alpha,\beta} := \begin{bmatrix} x^1(0) & x^1(0) + y_1^1 & \ldots & x^1(0) + y_p^1 \\ \vdots & \vdots & & \vdots \\ x^i(0) & x^i(0) + y_1^i & \ldots & x^i(0) + y_p^i \\ \vdots & \vdots & & \vdots \\ x^L(0) & x^L(0) + y_1^L & \ldots & x^L(0) + y_p^L \end{bmatrix}.$$

Remark 5 Let

$$s_p^i : [0, p] \to \left[x^i(0) + a, x^i(0) + b\right] \quad (i = 1, ..., L),$$

the piecewise linear functions whose node-sets are $N_p^i := \{(k, x^i(0) + y_k^i) : k = 0, 1, ..., p\}$ $(i = 1, ..., L)$. We denote by $S(E_p)$ the set of such functions. Let $i \in \{1, ..., L\}$ and let T_k^i $(k = 1, ..., p)$ be the triangle of vertices $\left(k-1, x^i(0) + y_{k-1}^i\right)$, $\left(k, x^i(0) + y_{k-1}^i + \frac{a}{p}\right)$

and $\left(k, x^i(0) + y^i_{k-1} + \frac{b}{p}\right)$. Then it is easy to verify that the following *geometric condition* holds:

$$\left\{ \left(x, s^i_p\left(x\right)\right) : x \in [k-1, k] \right\} \subseteq T^i_k \quad (k = 1, ..., p).$$

Step 3. Let E_p be an experiment of size L and length $p+1$ and let $SFD_{\alpha,\beta}$ be its matrix of simulated functional data. We set

$$m_{FD} = \min_{i=1,\dots,L} \left\{x^i(T)\right\}, \quad M_{FD} = \max_{i=1,\dots,L} \left\{x^i(T)\right\}$$

and

$$m_{SFD_{\alpha,\beta}} = \min_{i=1,\dots,L} \left\{x^i\left(0\right) + y^i_p\right\}, \quad M_{SFD_{\alpha,\beta}} = \max_{i=1,\dots,L} \left\{x^i\left(0\right) + y^i_p\right\}.$$

In order to compare the functional data FD with the simulated functional data $SFD_{\alpha,\beta}$ we first use the linear transformation

$$\phi_{FD}(u) = \frac{1}{M_{FD} - m_{FD}}(u - m_{FD}), \ u \in [m_{FD}, M_{FD},]$$

and the subdivision in K classes $\left[0, \frac{1}{K}\right[, \dots, \left[\frac{K-1}{K}, 1\right]$ of $[0, 1]$ to obtain an histogram $\mathcal{I}_{FD}$ of frequency distribution of data $\left\{\phi_{FD}(x^1(T)), \dots, \phi_{FD}(x^L(T))\right\}$. Then we use the linear transformation

$$\phi_{SFD_{\alpha,\beta}}(u) = \frac{1}{M_{SFD_{\alpha,\beta}} - m_{SFD_{\alpha,\beta}}}(u - m_{SFD_{\alpha,\beta}}), \ u \in \left[m_{SFD_{\alpha,\beta}}, M_{SFD_{\alpha,\beta}}\right]$$

and the same subdivision to obtain the histogram $\mathcal{I}_{SFD_{\alpha,\beta}}$ of frequency distribution of the data $\left\{\phi_{SFD_{\alpha,\beta}}(x^i\left(0\right) + y^i_p), \dots, \phi_{SFD_{\alpha,\beta}}(x^L(T))\right\}$. We denote by $E_{FD}(j)$, $E_{SFD_{\alpha,\beta}}(j)$ the frequencies of the classes $\left[\frac{j-1}{K}, \frac{j}{K}\right[$ $(j = 1, \dots, K-1)$, and by $E_{FD}(K)$, $E_{SFD_{\alpha,\beta}}(K)$ the frequencies of the class $\left[\frac{K-1}{K}, 1\right]$ of the above data, respectively.

Definition 6 *Let*

$$E_p = \left\{(x^i(0), x^i(0) + y^i_1, ..., x^i(0) + y^i_p), \ i = 1, ..., L\right\}$$

be an experiment of size L and length $p+1$. One of its proofs $(x^i(0), x^i(0) + y_1^i, ..., x^i(0) + y_p^i)$ *is assumed bad (good) if*

$$x^i(0) + y_p^i < \frac{min(G) + max(B)}{2} \quad \left(x^i(0) + y_p^i \geq \frac{min(G) + max(B)}{2}\right).$$

In the following we denote by Y_{E_p} the vector of outcomes of the experiment E_p according to the previous definition:

$$Y_{E_p} := \begin{bmatrix} \rho^1 \\ \vdots \\ \rho^i \\ \vdots \\ \rho^L \end{bmatrix} \qquad (\rho^i \in \{bad, good\} \text{ for } i = 1, \ldots, L)$$

Definition 7 *Let $\eta > 0$, $\theta > 0$ be two fixed tolerances* [1]*. An experiment E_p of size L and length $p+1$ is called admissible if the following conditions hold:*

i) $\left|\frac{m_{FD} - m_{SFD_{\alpha,\beta}}}{m_{FD}}\right| \leq \eta$ *and* $\left|\frac{M_{FD} - M_{SFD_{\alpha,\beta}}}{M_{FD}}\right| \leq \eta$,

ii) $\sum_{j=1}^{K} \frac{(E_{FD}(j) - E_{SFD_{\alpha,\beta}}(j))^2}{E_{FD}(j)} \leq \theta$*, i.e. the chi-square value of $\mathcal{I}_{FD}$ and $\mathcal{I}_{SFD_{\alpha,\beta}}$ is less than or equal to θ.*

Remark 8 *Admissible experiments E_p can be obtained via a Monte-Carlo method based on the generated random pairs* $(\alpha, \beta) \in [10^{-1}, 10] \times [10^{-1}, 10]$.

Definition 9 *Let $\mathcal{E}_{\eta,\theta}$ be the set of all admissible experiments E_p of size L and length $p+1$. We define the stochastic process X as the set*

$$X = \bigcup_{E_p \in \mathcal{E}_{\eta,\theta}} S(E_p).$$

and the associated binary response

$$Y : X \to \{bad, good\}, \quad Y(s) = Y_{E_p}(s).$$

In the following we assume that the couple $(X, Y) \in \mathcal{S} \times \mathcal{R}$ simulates the continuous phenomenon from which data FD and R are observed.

[1] in Sec.5 we assume $\eta = 0.05$, $\theta = 5$

4 The adjustment curve

Let $(X, Y) \in \mathcal{S} \times \mathcal{R}$. In this section we introduce the notion of adjustment curve for the binary response Y of the process X.

Let $E_p \in \mathcal{E}_{\eta,\theta}$ be an admissible experiment. We start to introduce the adjustment curve $\gamma_{a,E_p} : [0,T] \to [0,1]$ for the binary outcome Y_{E_p} of the experiment E_p. We set

$$b_{E_p}(k) = \max\left\{x^i(0) + y_k^i : i = 1, \ldots, L \textit{ and } \rho^i = \textit{bad}\right\} \quad (k = 0, \ldots, p)$$

and

$$g_{E_p}(k) = \min\left\{x^i(0) + y_k^i : i = 1, \ldots, L \textit{ and } \rho^i = \textit{good}\right\} \quad (k = 0, \ldots, p).$$

Since $\min(G) > \max(B)$, by Definition 6 we observe that the sets

$$B_{E_p} := \left\{x^i(0) + y_p^i : i = 1, \ldots, L \textit{ and } \rho^i = \textit{bad}\right\}$$

and

$$G_{E_p} := \left\{x^i(0) + y_p^i : i = 1, \ldots, L \textit{ and } \rho^i = \textit{good}\right\}$$

are separated with $\min(G_{E_p}) > \max(B_{E_p})$. We consider the sets of nodes in $\mathbb{R}^2$

$$\mathcal{N}_{b_{E_p}} = \left\{\left(k, b_{E_p}(k)\right), k = 0, \ldots, p\right\}$$

and

$$\mathcal{N}_{g_{E_p}} = \left\{\left(k, g_{E_p}(k)\right), k = 0, \ldots, p\right\}.$$

Then we denote by $s_{b_{E_p}}$ and $s_{g_{E_p}}$ the real piecewise linear functions on $[0,p]$ which node sets are $\mathcal{N}_{b_{E_p}}$ and $\mathcal{N}_{g_{E_p}}$, respectively. Let

$$E_p = \left\{(x^i(0), x^i(0) + y_1^i, ..., x^i(0) + y_p^i),\ i = 1, ..., L\right\}$$

be an *experiment* of size L and length $p + 1$. We give the following:

Definition 10 *Let $i \in \{1, \ldots, L\}$ and $\rho^i = bad$ or $\rho^i = good$. The piecewise linear interpolant $s_p^i \in S(E_p)$ is called adjustable at the time $\tau \in [0, p]$ (for short $\tau-$adjustable) if there exists $k \geq \tau$ with $k \in \{0, 1, \ldots, p\}$ such that*

$$s_p^i(k) \geq g_{E_p}(k) \text{ or } s_p^i(k) \leq b_{E_p}(k).$$

Definition 11 *The adjustment curve $\gamma_{a,E_p} : [0, p] \to [0, 1]$ for the binary outcome Y_{E_p} of the experiment E_p is the function*

$$\gamma_{a,E_p}(\tau) = \frac{\left|\left\{s_p^i(\tau) : i = 1, \ldots, L \text{ and } s_p^i(\tau) \text{ is } \tau - adjustable\right\}\right|}{L} \quad (\tau \in [0, p])$$

Clearly $\left\{\gamma_{a,E_p} : E_p \in \mathcal{E}_{\eta,\theta}\right\} =: \{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ is a finite set. By the change of variable $\tau = \dfrac{p}{T} \cdot t$ $(t \in [0, T])$ we get, for every $E_p \in \mathcal{E}_{\eta,\theta}$,

$$\gamma_{a,E_p}(t) = \gamma_{a,E_p}(\frac{p}{T} \cdot t) \quad (t \in [0, T])$$

We consider the random experiment "*obtain an admissible experiment E_p*" whose sample space is the infinite set $\mathcal{E}_{\eta,\theta}$. Set $\mathcal{E}_{\eta,\theta}^i := \left\{E_p \in \mathcal{E}_{\eta,\theta} : \gamma_{a,E_p} = \gamma_i\right\}$ $(i = 1, \ldots, N)$. Assume that the numbers ν_i are the frequencies of the curves γ_i $(i = 1, \ldots, N)$.

Definition 12 *The adjustment curve $\gamma_a : [0, p] \to [0, 1]$ for the binary response Y of the process X is the function*

$$\gamma_a(t) = \sum_1^N \nu_i \gamma_i(t) \quad (t \in [0, T]).$$

Remark 13 In practice, given a couple $(X, Y) \in \mathcal{S} \times \mathcal{R}$ we can choose a tolerance $\epsilon > 0$ such that, if $E_p^1 = (x_1^1, \ldots, x_L^1)$, $E_p^2 = (x_1^2, \ldots, x_L^2)$ are two admissible experiments such that $\max_{i=1}^{L} \|x_i^1 - x_i^2\|_\infty \leq \epsilon$ (here $\|\cdot\|_\infty$ denotes the usual sup-norm) then E_p^1, E_p^2 can be considered indistinguishable. Therefore X becomes a process with a discrete number of realizations. Hence we can assume that for $i = 1, \ldots, N$ the frequency ν_i is computable in experimental way, i.e.

$$\nu_i = \lim_{n \to \infty} \nu_i^n,$$

where ν_i^n is the relative frequency of γ_i observed on a sample $(\gamma_1, \dots, \gamma_n)$ of size n.

Remark 14 Let $\gamma_a^n := \sum_1^n \nu_i^n \gamma_i$ $(n = 1, 2, \dots)$. It is not difficult to see that the sequence $\{\gamma_a^n\}$ converges to γ_a on $[0, T]$ and that the variance $Var(\gamma_a)$ of the random variable γ_a is less or equal 2. Therefore the classical Monte Carlo method can be used to produce approximations of γ_a with the needed precision.

5 Application

We present an application of our results to a real industrial process; namely we will show how our model can be used to monitoring and predicting the quality of a product resulting from a kneading industrial process. We will use a sample of data provided by Danone Vitapole Research Department (France)[2]. In kneading data from Danone, for a given flour, the resistance of dough is recorded during the first 480 seconds of the kneading process. There are 136 different flours and then 136 different curves or trajectories (functions of time). Each one of them is obtained by Danone as a mean curve of a number of replication of the kneading process for each different flour. Each curve is observed in 240 equispaced time points (the same for all flours) of the interval time $[0, 480]$. Depending on its quality, after kneading, the dough is processed to obtain cookies. For each flour the quality of the dough can be *adjustable*, *bad* or *good*. The sample contains 30 *adjustable*, 44 *bad* and 62 *good* observations (see Fig. 2). With an appropriate smoothing of Danone's data, i.e. without modifying the phenomenon in a significant way (see details in Remark 2) we reduced to a binary outcome R, $R \in \{bad, good\}$ with sets B and G separated (see Fig. 3). In Fig. 4 we show one admissible experiment E_p obtained by our

[2]We wish to thank the Danone Vitapole Research Department for kindly furnishing the data

method. In Fig. 5 is depicted the adjustment curve γ_a for the binary response Y of the stochastic process X related to the smoothed Danone's data. This curve has been computed on the basis of $n = 100$ admissible experiment E_p. In Fig. 6 we show the corresponding adjustment curves γ_{a,E_p} together with the adjustment curve $\gamma_{a,D}$ of Danone's data and the adjustment curve γ_a. We remark that in order to obtain by application of Monte Carlo Method the adjustment curve γ_a with and error less than 10^{-1} and probability greater than 90% we need to perform $n = 4000$ admissible experiments. As we pointed out in the introduction, the adjustment curve γ_a is an important tool for the analysis of a real process which evolves during a time period. In fact in the cookie's case, as shown in Fig. 5, the quality of the dough can be forecasted by the adjustment curve for each given $t \in [0, T]$ with increasing probability.

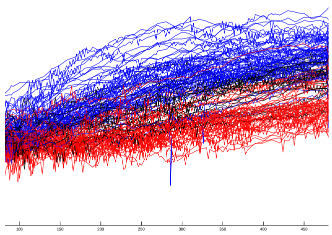

Figure 2: Danone's original data.

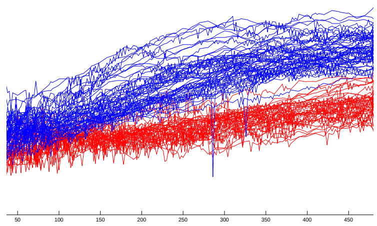

Figure 3: Danone's smoothed data.

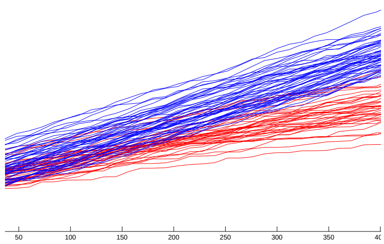

Figure 4: An admissible experiment E_p

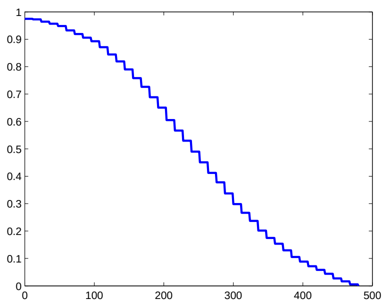

Figure 5: The adjustment curve γ_a.

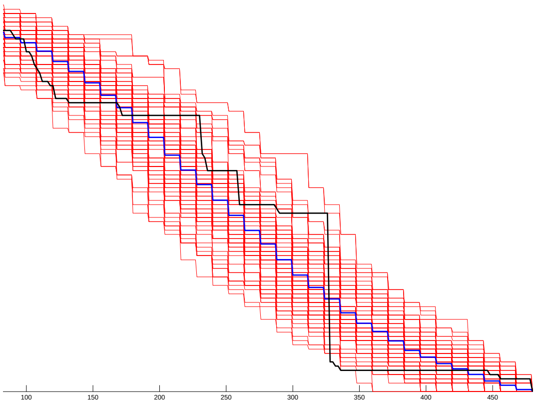

Figure 6: The adjustment curves γ_a (in blue), $\gamma_{a,D}$ (in dark) and γ_{a,E_p} (in red)

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